

Morse complexes for manifolds with nonempty boundary and A_∞ -structures.

Applications to links in S^3 .

I Example $h: S^3 \rightarrow \mathbb{R}$ height function

$L \subset S^3$ link

$$M = S^3 \setminus \text{nbd}(L) . f = h|_M$$

II Generic Morse function, $M^n \xrightarrow{f} \mathbb{R}$, $\partial M \neq \emptyset$

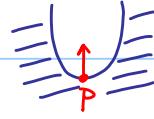
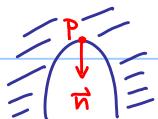
$\text{crit } f$ nondeg. generic means $\text{crit } f \cap \partial M = \emptyset$

$f_\partial := f|_{\partial M}$ is Morse

$$\text{crit } f_\partial = \text{crit}^N \sqcup \text{crit}^D$$

Neumann

Dirichlet



$$\langle df(p), \vec{n} \rangle < 0$$

$$\langle df(p), \vec{n} \rangle > 0$$

$$\text{Modules: } C_k^N(f) = \mathbb{Z} \langle \text{crit}_k f, \text{crit}_{k-1}^N f \rangle$$

$$C_k^D(f) = \mathbb{Z} \langle \text{crit}_k f, \text{crit}_{k-1}^N f \rangle$$

Thm: 1) $C_*^N(f)$ can be equipped with d^N s.t. $H_k(C_*^N, d^N) \cong H_k(M; \mathbb{Z})$

2) $C_*^D(f)$ can be equipped with d^D s.t. $H_k(C_*^D, d^D) \cong H_k(M, \partial M; \mathbb{Z}^\times)$

3) Short exact sequence

↑
twisted coeff.

4) These complexes have A_∞ -str (A_∞ -alg)

1) & 2) Geometriae Dedicata 2011

3) & 4) With C Blanchet.

III History

For closed manifolds

Morse complex appeared in S. Smale thm. (Annals 1961)

$C(f, X)$

↑
pseudo-gradient vector field satisfying Morse condition

E. Witten (Supersymmetry and Morse Theory, JDG 1982)

$$d_{\text{Witten}} = e^{-f/h} d_{DR} e^{f/h} \quad \text{w.r.t. } \Delta_{\text{Witten}}$$

Riem. metric

the diff. forms which eigenvectors with "small" eigenvalues generate a Morse complex.

worked out by Helffer - Sjöstrand.

For manifolds with $\partial M \neq \emptyset$, same programme works

Chang - Liu, 1995

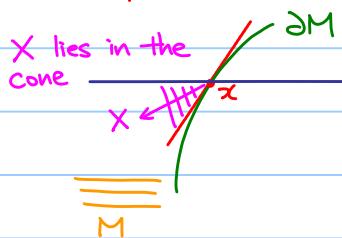
Helffer - Nier, 2006

Construction of pseudo-gradient. adapted

Question : Find a vector field s.t.

- $X \cdot f < 0$ out of $\text{crit } f$ and $\text{crit } f_s$.
- X pointing inwards.

local problem!



Obstruction : the cone has empty interior

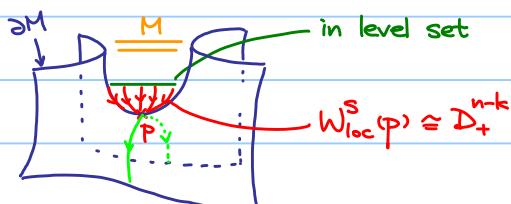
Def: X^N is an adapted pseudo-gradient if

- $X^N \cdot f < 0$ out of $\text{crit } f$ and $\text{crit } f_s$.
- X^N is tangent to ∂M near $\text{crit}^N f_s$
- X^N is pointing in in the complement of these nbd.
- $X^N \cdot f$ is a Morse.

Observe that X^N is positively complete

- $W^u(p)$ of dim k , $\cong \mathbb{R}^k$, if $p \in \text{crit}_k f$, $\text{crit}_k f_s$
- $W_{\text{loc}}^s(p)$ exists

Assume X^N is Morse-Smale



$$\langle d^N x, y \rangle = \sum \# \text{lines in } (W^u(x) \cap W_{\text{loc}}^s(y))$$

\uparrow
crit pt. of index $k-1$

proof of 2: Look at $-f$, $\text{crit}_k f = \text{crit}_{n-k}(-f)$

$$\text{crit}_{k-1}^D f_s = \text{crit}_{n-k}^N (-f)$$

Choose X^D an adapted pseudo-gradient for $(-f)$

$$C_D^k(f, X^D) = C_{n-k}^N(-f, X^D)$$

$$H^k(C_*^D(f, X^D)) \underset{\text{is}}{=} H_{n-k}(C_*^N(-f, X^D))$$

$$H^k(M, \partial M; \mathbb{Z}^{\text{or}}) \underset{\text{is}}{=} H_{n-k}(M; \mathbb{Z})$$

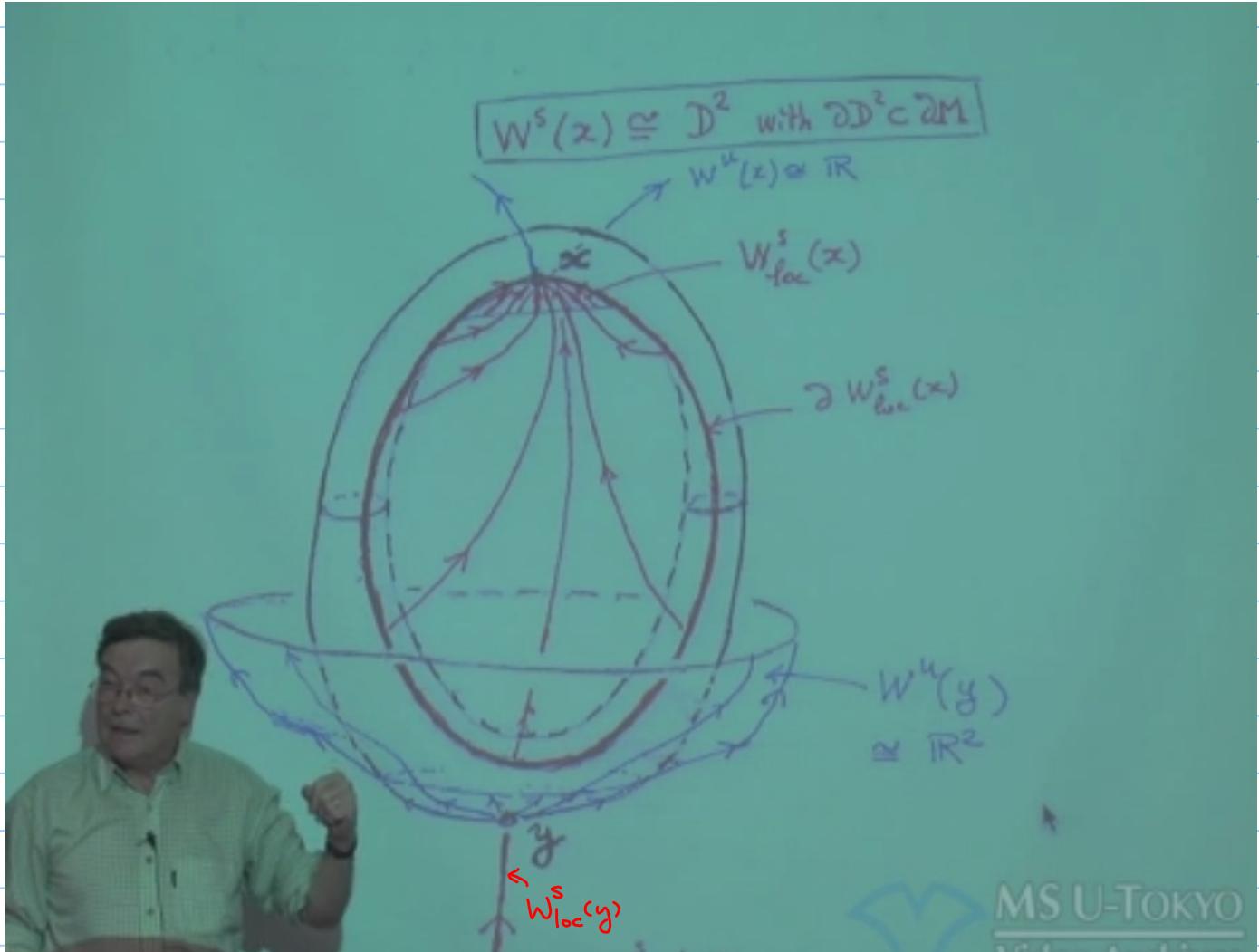
Define $C_k^D(f, X^D) = \text{Hom}(C_D^k(f, X^D), \mathbb{Z})$

Since C_*^D has finite type, $H_*(f, X^D) \cong H_*(M, \partial M; \mathbb{Z}^{\text{or}})$

$$C_*^D \otimes \mathbb{Z}^{\text{or}} \rightarrow H_*(M, \partial M; \mathbb{Z})$$

Actually, there exists $W^s(x) = \bigcup_{t>0} \phi_t^{-1}(W_{\text{loc}}^s(x))$

e.g. 2



e.g. 2

